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The effect of a stationary windfield on a rotating shallow sea

by

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The effect of a stationary windfield on a rotating shallow sea [1)]

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§1. Introduction, General equations.

As a part of the research on the hydrodynamical behaviour of a shallow sea under influence of storm surges (see the foregoing paper of H.A. Lauwerier), the case of a stationary windfield has been considered in some detail (VELTKAMP, 1953,1954).

We start with the following differential equations

$$\left.\begin{aligned} \lambda u - \Omega v + g\frac{\partial \zeta}{\partial x} &= (\rho h)^{-1}\, U \\ \lambda v + \Omega u + g\frac{\partial \zeta}{\partial y} &= (\rho h)^{-1}\, V, \end{aligned}\right\} \tag{1.1}$$

$$\frac{\partial}{\partial x}(hu) + \frac{\partial}{\partial y}(hv) = 0. \tag{1.2}$$

where u,v are averages over a vertical of the horizontal components of the velocity, ζ the elevation of the sealevel above the undisturbed level, U,V the components of the tangential stress on the surface of the sea due to the wind, h the depth of the sea (in undisturbed position), and λ, Ω, g and ρ coefficients of friction, Coriolisforce, gravity and density, respectively.
In the derivation of these equations considerable simplifications have been introduced: linearisation, neglect of vertical motions, a simple model for the turbulent friction, etc. [2)]

In this paper we shall only consider the case that all coefficients λ, Ω, g, ρ and h are independent of x and y. In this case a rather elegant treatment with the aid of function-theoretic methods is possible.

It is well-known that any two-dimensional vector-field (U,V) can be separated into an irrotational and a solenoidal field, each of which being generated by a scalar potential:

$$\left.\begin{aligned} U &= -\rho g h\frac{\partial \Psi}{\partial x} - \rho g h \frac{\partial \Phi}{\partial y} \\ V &= -\rho g h\frac{\partial \Psi}{\partial y} + \rho g h \frac{\partial \Phi}{\partial x} \end{aligned}\right\} \tag{1.3}$$

Furthermore, on behalf of (1.2) we can introduce a stream function $\theta(x,y)$ by

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1) Research carried out under the direction of Prof.Dr D. van Dantzig.

2) A careful discussion of these matters has been given by SCHÖNFELD (1954). Compare also SCHALKWIJK (1947).

$$u = -(g/\lambda)\frac{\partial\theta}{\partial y}, \quad v = (g/\lambda)\frac{\partial\Theta}{\partial x}. \tag{1.4}$$

Substitution of (1.3) and (1.4) into (1.1) gives

$$\begin{aligned} \frac{\partial}{\partial x}(\Theta - \Phi) &= \frac{\partial}{\partial y}\left(\frac{\Omega}{\lambda}\Theta - \zeta - \Psi\right) \\ \frac{\partial}{\partial y}(\Theta - \Phi) &= -\frac{\partial}{\partial x}\left(\frac{\Omega}{\lambda}\Theta - \zeta - \Psi\right) \end{aligned} \tag{1.5}$$

These equations are directly seen to be the Cauchy-Riemann differential equations. Accordingly an <u>analytic</u> function $Z(z) = X(x,y) + i\,Y(x,y)$ of $z=x+iy$ must exist such that

$$\Theta - \Phi = X, \quad \frac{\Omega}{\lambda}\Theta - \zeta - \Psi = Y,$$

or

$$\Theta = X + \Phi, \quad \zeta = \frac{\Omega}{\lambda}(X + \Phi) - (Y + \Psi) \tag{1.6}$$

Hence, if Ψ, Φ and Z are known, the problem is completely solved.

The function $Z(z)$ can be determined with the aid of <u>boundary-conditions</u>.

Let the sea be represented by a domain D with boundary Γ. On parts of Γ which represent <u>coasts</u>, the normal components of the mean velocity (u,v) must vanish, hence, if we assume D to be simply-connected, we have the boundary-condition $\Theta=0$ or $\operatorname{Re} Z = -\Phi$. (1.7)

On parts of Γ which represent the connection of the sea with a deep ocean (the level of which is assumed to be undisturbed), we have $\zeta=0$, or $\frac{\Omega}{\lambda}\operatorname{Re} Z - \operatorname{Im} Z = -\frac{\Omega}{\lambda}\Phi + \Psi$, or, defining the real number μ by

$$\operatorname{tg}\pi\mu = (\Omega/\lambda), \quad -\tfrac{1}{2} < \mu < \tfrac{1}{2}, \tag{1.8}$$

$$\operatorname{Im}(e^{-\pi i\mu} Z) = \Phi\sin\pi\mu - \Psi\cos\pi\mu. \tag{1.9}$$

§2. <u>Shallow sea bounded by coasts only.</u>

If the wind-forces (U,V) are given, the potentials Ψ and Φ are of course not uniquely determined by (1.3). A special pair of potentials can be found as follows. Let $\Phi_o(x,y)$ and $\Psi_o(x,y)$ be determined by the conditions

$$\left.\begin{aligned} \nabla^2\Phi_o &= -(\varrho gh)^{-1}\left(\frac{\partial U}{\partial y} - \frac{\partial V}{\partial x}\right) \text{ in } D \\ \Phi_o &= 0 \quad \text{on } \Gamma \end{aligned}\right\} \tag{2.1}$$

$$\left.\begin{aligned} \nabla^2\Psi_o &= -(\varrho gh)^{-1}\left(\frac{\partial U}{\partial x} + \frac{\partial V}{\partial y}\right) \text{ in } D \\ \frac{\partial\Psi_o}{\partial n} &= -(\varrho gh)^{-1}\, W_n \quad \text{on } \Gamma \end{aligned}\right\} \tag{2.2}$$

where W_n is the normal component of (U,V) on Γ. Then Φ_o is uniquely determined and Ψ_o is uniquely determined apart from an additive constant. Furthermore it is directly seen that the vector $U + \rho gh\left(\frac{\partial \Psi_o}{\partial x} + \frac{\partial \Phi_o}{\partial y}\right)$, $V + \rho gh\left(\frac{\partial \Psi_o}{\partial y} - \frac{\partial \Phi_o}{\partial x}\right)$ is zero in D since it is both irrotational and solenoidal and its normal component on Γ is zero. Accordingly, the vector field U,V is generated by the potentials Ψ_o and Φ_o by means of (1.3).

For a sea, bounded by coasts only, (1.7) now gives Re Z $= -\Phi_o = 0$ on Γ, hence we have in D $Z(z) = iC$, where C is a real constant. Accordingly

$$\Theta = \Phi_o, \quad \zeta = \frac{\Omega}{\lambda}\Phi_o - \Psi_o - C \tag{2.3}$$

Let us assume that the free constant in Ψ_o has been determined such that $\iint_D \Psi_o \, dS = 0$. Then, since the total amount of water in the sea must be constant, we have

$$0 = \iint_D \zeta \, dS = \frac{\Omega}{\lambda}\iint_D \Phi_o \, dS - C.\iint_D dS,$$

or

$$C = \frac{\Omega}{\lambda}\overline{\Phi}_o,$$

where $\overline{\Phi}_o$ is the mean value of Φ_o over D. Accordingly, we have the unique solution

$$\zeta = \frac{\Omega}{\lambda}(\Phi_o - \overline{\Phi}_o) - \Psi_o \tag{2.4}$$

This formula permits some interesting conclusions.

a. In a non-rotating sea, the elevation of the surface is independent of the solenoidal part of the wind forces.

b. The difference between the elevations at the coast of a rotating and a non-rotating sea is a constant along the coast, this constant is proportional to Ω/λ and only dependent on the solenoidal part of the wind force.

c. In an actual storm-surge, caused by a depression on the Northern hemisphere the curl $\frac{\partial V}{\partial x} - \frac{\partial U}{\partial y}$ of the wind force is usually positive and accordingly $\overline{\Phi}_o$ is negative ($\Phi_o = 0$ on Γ and $\nabla^2 \Phi_o > 0$ in D). Consequently the presence of Coriolis force results in an increase of elevation at the coasts.

<u>Remark.</u>

If potentials Φ and Ψ are given which do not satisfy the boundary conditions in (2.1) and (2.2), one finds the corresponding potentials Φ_o and Ψ_o as follows. Let $Z(z) = X + iY$ be an analytic function of z in D such that on Γ $X = \text{Re } Z = -\Phi$ (Z is uniquely determined up to an additive imaginary constant). Then $\Phi_o = \Phi + X$, $\Psi_o = \Psi + Y$.

§3. Shallow sea connected with a deep ocean.

Let us consider the case that the boundary Γ consists of two parts Γ_1 and Γ_2, where Γ_1 represents the connection with a deep ocean and Γ_2 represents a coast. If the wind-forces are represented by potentials Φ_o and Ψ_o which satisfy the boundary-conditions of (2.1) and (2.2), respectively, we have for the analytic function Z(z), introduced in §1, the following boundary-conditions (compare (1.7) and (1.9))

on Γ_1: $\mathrm{Im}\,(e^{-\pi i \mu} Z) = -\Psi_o \cos \pi\mu$,

on Γ_2: $\mathrm{Re}\, Z = 0$.

We shall now prove the following formula:
Let w(z) be an analytic function of z in D which maps D conformally on the half-plane $\mathrm{Im}\, w > 0$ such that Γ_1 is mapped on the real segment $\mathrm{Im}\, w = 0$, $|\mathrm{Re}\, w| < 1$ and Γ_2 on the segments $\mathrm{Im}\, w=0$, $|\mathrm{Re}\, w| > 1$ [3]. Then we have

$$Z(z) = -\cos\pi\mu \,.\, (w(z)-1)^{\mu+\frac{1}{2}}\,(w(z)+1)^{-\mu+\frac{1}{2}} \,.$$

$$.\ \frac{1}{\pi i}\int_{\Gamma_1} (1-w(t))^{-\mu-\frac{1}{2}}(1+w(t))^{\mu-\frac{1}{2}}\,\frac{\Psi_o(t)w'(t)}{w(t)-w(z)}\,dt. \qquad (3.1)$$

Proof:

It is not difficult to see that if $\mathrm{Im}\, w=0$, $|\mathrm{Re}\, w| > 1$, $(w-1)^{\mu+\frac{1}{2}}(w+1)^{-\mu+\frac{1}{2}}$ is real. And since clearly the integral is real if z is on Γ_2, the boundary-condition on Γ_2 is satisfied.

If z tends to a point z_o on Γ_1 (from the side of D), we have by application of a formula of PLEMELJ (cf MUSKHELISHVILI, 1953, Ch.2)[4]

$$\mathrm{Re}\,\lim_{z\to z_o}\frac{1}{\pi i}\int_{\Gamma_1}(1-w(t))^{-\mu-\frac{1}{2}}\,(1+w(t))^{\mu-\frac{1}{2}}\,\frac{\Psi_o(t)w'(t)}{w(t)-w(z)}\,dt =$$

$$= (1-w(z_o))^{-\mu-\frac{1}{2}}\,(1+w(z_o))^{\mu-\frac{1}{2}}\,\Psi(z_o). \qquad (3.2)$$

Furthermore, if z is on Γ_1, 1-w and 1+w are positive and $w-1= e^{\pi i}(1-w)$, so $(w-1)^{\mu+\frac{1}{2}}(w+1)^{-\mu+\frac{1}{2}} = i\, e^{\pi i\mu}\,(1-w)^{\mu+\frac{1}{2}}(1+w)^{-\mu+\frac{1}{2}}$ (3.3)

Combination of (3.2) and (3.3) now shows that the boundary-condition on Γ_1 is satisfied.

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3) w(z) is not uniquely determined by these conditions, but that does not matter, however.

4) This formula reads $\lim_{z\to z_o}\frac{1}{\pi i}\int_{\Gamma_1}\frac{\varphi(t)}{t-z}\,dt = \varphi(z_o) + \frac{1}{\pi i}\oint_{\Gamma_1}\frac{\varphi(t)}{t-z_o}\,dt$, where the second integral is a Cauchy-integral. Hence, if $\varphi(z)$ is real (on Γ_1), Re lim....= $\varphi(z_o)$.

Finally, $Z(z)$ is clearly analytic in D and from theorems cited by MUSKHELISHVILI (l.c. Chs.2 and 4) it follows that $Z(z)$ is continuous in $D+\Gamma$. This completes the proof of the validity of formula (3.1).

As has been pointed out in § 1, the knowledge of Z (and Φ and Ψ) implies the complete solution of the problem.

4. Rectangular sea connected with an ocean.

Let D be the domain $-a < x < a$, $0 < y < b$ and let the part $y=b$, $-a < x < a$ of Γ be the connection of the sea with a deep ocean, the other parts of Γ being coasts. This domain can be considered as an approximate model for the North Sea, cf. fig.1.

For the mapping function $w(z)$, introduced in § 3, one has

$$w(z) = -\,\mathrm{sn}\,\left(\frac{z-ib}{a}K,k\right). \tag{4.1}$$

where the modulus K of the Jacobian elliptic function follows from the condition

$$\frac{K'(k)}{K(k)} = \frac{b}{a}$$

($K(k)$ being the complete elliptic integral of the first kind and $K'(k)=K(\sqrt{1-k^2})$).

If k is small (which is the case for $b \gg a$, for the North Sea we have roughly $b=4a$ and accordingly $k \sim 4e^{-(\pi b/2a)}=0{,}747.10^{-2}$), one has on Γ_1, where $z=x+ib$, x real,

$$\mathrm{sn}\,\left(\frac{Kx}{a},k\right) \sim \sin\frac{\pi x}{2a}\,. \tag{4.2}$$

For $z=0$ we have $w=\infty$ and accordingly one finds from (3.1), (4.1) and (4.2) after some algebra

$$Z(0) = -\frac{i}{2a}\cos\pi\mu\int_{-a}^{a}\mathrm{tg}\,\left(\frac{\pi(t+a)}{4a}\right)^{-2\mu}\Psi_0(t,b)\,dt$$

and this constant can be shown to be a satisfactory approximation for $Z(z)$ for all values of z near the „South coast" of the sea.

Let us now consider a specific wind field:

$$U = U_0 + U_1x + U_2(y-\tfrac{1}{2}b)$$
$$V = V_0 + V_1x + V_2(y-\tfrac{1}{2}b)$$

One then finds for the value of the elevation ζ at the middle of the south coast

a. if $\Omega=0$:
$$\rho gh\,\zeta(0,0) = -b\,V_0 - \frac{1}{6}a^2U_1$$

b. if $\Omega/\lambda = 5{,}3$ [5]:

$$\rho gh\, \zeta(0,0) = -\, b\, V_o - 0{,}410a^2 U_1 + 0{,}845a(U_o + \tfrac{1}{2}b\, U_2) + $$
$$+ 0{,}472a^2(V_1 - U_2)$$

Here the first term $-b\, V_o$ is the main term, due to the average of the axial component of the mind. The last term is due to the curl of the wind-forces, whereas the second and third terms are due to the x-component of the wind-forces near the entrance of the sea.

If for instance we have $a = \frac{1}{4}b$ and $U=0$, $V=-V_o(1-\frac{x}{a})$, we find

$$\zeta(0,0) = 1{,}12\, \frac{bV_o}{\rho gh}\,,$$

hence a 12%-increase over the case of no Coriolis-force.

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5) which seems to be a reasonable approximation, cf. SCHALKWIJK, 1956, p.21.